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An Exploration of the Irrational Nature of the Natural Exponent e

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Abstract

The rational or irrational property of real numbers is an arithmetic property, and the rational or irrational characteristics of some important constants are closely related to the properties of integers and the distribution of prime numbers. The natural exponent e as a well-known irrational number has attracted close attention of mathematicians. We first investigate how to construct suitable auxiliary functions for e such that the proof of its irrational properties can be described uniformly. We give a key auxiliary function and construct the exponential function based on the difference of the properties of e itself, and accordingly we obtain some inferences that the irrational properties of e and its idempotent forms can be proved uniformly.

Keywords

Natural exponent e; Irrational properties.

1. Introduction

The rational or irrational nature of real numbers is an arithmetic property, so it is not surprising to encounter important constants [1-6], whose rational or irrational nature is related to the nature of integers and the distribution of primes [7, 8], such as the number $6/\pi^2 = \prod_{p>2}(1-1/p^2)$.

Moreover, mathematics is an extremely rigorous science, and to assert that both numbers are irrational would require giving proof. $\sqrt{2}$ is the first irrational number ever discovered by man, and the method and procedure of its proof are relatively concise and clear. The proof that e is irrational number is a bit more complicated. In this paper, we want to find out how to prove that e is irrational number.

2. Basic Concepts and Preparatory Knowledge

This section describes the basic concepts, notations, and preparatory knowledge used throughout the work.

2.1. Rational and Irrational Criteria

If $\alpha = a/b$, where $a,b \in \mathbb{Z}$ are integers, then the number $\alpha \in \mathbb{R}$ is called rational. Otherwise, the number is irrational. Irrational numbers can be classified as algebraic and transcendental numbers. α is algebraic if it is a root of an irreducible polynomial $f(x) \in \mathbb{Z}[x]$ with number $\deg(f) > 1$ and vice versa [9].

Lemma 1 (Rational Criterion) If a real number $\alpha \in \mathbb{Q}$ is a rational number, then there exists a constant $c = c(\alpha)$ such that

$$\frac{c}{q} \le \left| \alpha - \frac{p}{q} \right| \tag{1}$$

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holds for any rational fraction $p/q \neq \alpha$. Specifically, if $\alpha = \alpha/B$ then $c \geq 1/B$.

This is a mathematical expression about the difficulty of any rational number $\alpha \in \mathrm{Q}$ being effectively approximated by other rational numbers [10-12]. On the other hand, the irrational number $\alpha \in \mathrm{R} - \mathrm{Q}$ can be effectively approximated by rational numbers. If the inequality $|\alpha - p/q| < c/q$ complementary to Equation 1 holds approximately for an infinite number of rational numbers p/q, then it is sufficiently clear that the real number $\alpha \in \mathrm{R}$ is irrational.

Lemma 2 (Irrational Criterion) Let $\psi(x)=o(1/x)$ be a monotonically decreasing function, such that $\alpha \in \mathrm{Q}$ is a real number, if

$$0 < \left|\alpha - \frac{p}{q}\right| < \psi(q) \tag{2}$$

holds for infinitely many rational fractions $p/q \in \mathrm{Q}$, then α is irrational [10-12].

Proof: by Lemma 1 and assumptions, it follows that

$$\frac{c}{q} \leq \left|\alpha - \frac{p}{q}\right| < \psi(q)=o(\frac{1}{q}) \tag{3}$$

However, this is a contradiction because $c/q \neq o(1/q)$. A more precise theorem for testing that any real number is irrational is discussed below.

Theorem 1 Suppose aR is an irrational number, then there exists an infinite sequence of rational numbers p_n/q_n satisfying

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2} \tag{4}$$

holds for any integer $n \in \mathrm{N}$ [10-12].

For a continuous fraction $a_i \geq a > 1$ of the larger term $\alpha = [a_0, a_1, a_2, \mathrm{B}\ ]$, where a is a constant, there is a slightly better inequality.

Theorem 2 Let $[a_0, a_1, a_2, \ldots]$ be a sequence of continuous fractions $\{p_n/q_n : n \geq 1\}$ of real numbers that are convergent, then there is.

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{a_n q_n^2} \tag{5}$$

holds for any integer $n \in \mathrm{N}$ [10-12].

This is a standard mathematical formulation in the literature [10-12], and related proofs appear in similar references [13-15]. A theorem that provides a more general application to almost all real inequalities is as follows.

Theorem 3 Let ψ be a monotonically decreasing real function, $\alpha \in \mathrm{R}$. If there exists an infinite sequence of rational approximations p_n/q_n such that $p_n/q_n \neq \alpha$ and.

$$0 < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{\psi(q_n)}{q_n} \tag{6}$$

and $\sum_q \psi(q) < \infty$ then the real numbers α are approximable to ψ.

2.2. A Key Helper Function

Construct the auxiliary function.

$f(x) = \frac{x^n(1-x)^n}{n!}$, and prove that this function satisfies the following three properties.

Property I $f(x)$ is a polynomial of form $\sum_{i=n}^{2n} \frac{c_i}{i!} x^i$ and satisfies that the coefficients c_i are all integers.

Property II When $0 < x < 1$, $0 < f(x) < \frac{1}{n!}$.

Property III For all integers $m \geq 0$, the m-th order derivatives of $f(x)$ must have integer values at 0 and 1, i.e., $f^{(m)}(0)$ and $f^{(m)}(1)$ are also integers.

Property I and Property II are obviously valid, and Property III is proved below. $f(x)$ is a sum of $n+1$ terms from the n th power of x to the $2n$ th power of x, according to Property I. Therefore, when $m < n$, $f^{(m)}(0)$ is 0, which is of course an integer, and when $m > 2n$, $f^{(m)}(x)$ is constantly 0, which is $f^{(m)}(0)$, of course, also an integer.

And when $n \leq m \leq 2n$, the m th order derivative of $f(x)$ according to the polynomial of property I yields $f^{(m)}(0) = \frac{c_m \cdot m!}{n!}$, and since c_m is an integer and $m \geq n$, this number must be an integer. Therefore $f^{(m)}(0)$ must be an integer. Also, notice that this function has a very obvious symmetry, i.e.

$$f(x) = f(1-x) \tag{7}$$

Taking the derivative of order m for both sides of this equation at the same time, and after that we get.

$$f^{(m)}(x) = (-1)^m f^{(m)}(1-x) \tag{8}$$

from which we have $f^{(m)}(0) = (-1)^m f^{(m)}(1)$, so since $f^{(m)}(0)$ is an integer, then $f^{(m)}(1)$ is also an integer and Property III holds.

3. Proof of Key Conclusions

3.1. Proof That E Is An Irrational Number

We start with the simplest problem, proving that e is irrational [16-18]. If the function e^x is subjected to a Taylor series expansion at the point $x = 0$, and then after substituting $x = 1$ into the resulting infinite term series expansion, the following well-known formula is obtained, i.e.

$$e=\sum_{i=0}^{\infty}\frac{1}{i!}=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+........ \tag{9}$$

Without resorting to Taylor series expansions, one can also use the following approach to give a less rigorous proof of the above equation from the definition of e.

We know that by definition, $e=\lim_{n\to\infty}(1+\frac{1}{n})^n$, and let us first look at the expression $(1+\frac{1}{n})^n$ for the limit being sought, expanding this power expression according to the binomial decomposition as follows.

$$\begin{aligned}(1+\frac{1}{n})^n&=C_n^0+C_n^1\cdot\frac{1}{n}+C_n^2\cdot\frac{1}{n^2}+C_n^3\cdot\frac{1}{n^3}+.......+C_n^n\cdot\frac{1}{n^n}=1+\frac{n}{1!\cdot n}+\frac{n(n-1)}{2!\cdot n^2}\\&+\frac{n(n-1)(n-2)}{3!\cdot n^3}+.......+\frac{n(n-1)(n-2)......(n-i+1)}{i!\cdot n^i}+.......\frac{n(n-1)(n-2)......2\cdot 1}{n!\cdot n^n}\end{aligned} \tag{10}$$

We observe the i-th term of which (note that i here is independent of n) and set the i-th term to a_i, with

$$\frac{1}{i!}\cdot(\frac{n-i+1}{n})^i<a_i=\frac{n(n-1)(n-2)......(n-i+1)}{i!\cdot n^i}<\frac{1}{i!} \tag{11}$$

It is obvious that both sides of Equation 11 converge to $\frac{1}{i!}$ as n tends to $+\infty$, so apply the pinch-force theorem for the limit.

$$\lim_{i\to\infty}a_i=\frac{1}{i!} \tag{12}$$

Since $(1+\frac{1}{n})^n$ expands to an n-term sum, when n tends to $+\infty$, it obviously becomes an infinite term sum. For n tending to $+\infty$, each obtained a_i corresponds to a specific, finite i. As a result of the above derivation, any specific a_i is equal to $\frac{1}{i!}$ and the resulting infinite sum of terms must be,

$$e=1+\frac{1}{1!}+\frac{1}{2!}+........ \tag{13}$$

Thus.

$$e=\lim_{n\to\infty}(1+\frac{1}{n})^n=1+\frac{1}{1!}+\frac{1}{2!}+........=\sum_{i=0}^{\infty}\frac{1}{i!} \tag{14}$$

After proving the above conclusion, the process, and method of proving that e is an irrational number is simpler. We apply the converse method to prove that e is indeed an irrational number.

Assuming that e is a rational number, we may set $e = \frac{a}{b}$, where a and b are positive integers, and we then take a positive integer n and multiply both sides of this equation by $b \cdot n!$ to get.

$$b \cdot n! \cdot e = a \cdot n! \tag{15}$$

Obviously, the right side of equation 15 is an integer, while its left side is.

$$\begin{aligned} b \cdot n! \cdot e &= b \cdot n! \cdot (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \ldots\ldots) \\ &= b \cdot n! \cdot (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \ldots\ldots + \frac{1}{n!}) + b \cdot (\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \ldots\ldots) \end{aligned} \tag{16}$$

The first term of this equation is clearly an integer, yet the second term is clearly faulty so that the second term is equal to M. We have.

$$\begin{aligned} 0 < M &= b \cdot (\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \ldots\ldots) \\ &< b \cdot (\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \ldots\ldots) = \frac{b}{n} \end{aligned} \tag{17}$$

Since it is n we arbitrarily choose a positive integer, as long as we get the value of n large enough so that $n > b$, we get $0 < M < 1$, thus making it impossible for M to be an integer. Thus the left side of equation 15 is not an integer, while its right side must be an integer, a contradiction.

Thus e cannot be a rational number, and the proof is over.

3.2. Proof That e^k (k Is A Positive Integer) Is Irrational

If we are familiar with the Taylor series expansion of e^x, we can use the previous method to prove that e is irrational in a similar way to prove that e^2 is also irrational. First assume that $e^2 = \frac{a}{b}$, and then get $b \cdot e = a \cdot e^{-1}$, while using the Taylor series expansion of e and e^{-1}, and find that one of the two sides of the equation is a little larger than some integer and the other side is a little smaller than some integer, in which case the two numbers cannot be equal, thus deriving a contradiction. No further details will be elaborated here [19-22].

Also, thinking a bit more, we can see that studying whether e^k is an irrational number is a relatively meaningful problem. If for any positive integer k, there is for irrational e^k, so that any rational power we obtain is easily irrational, this is because: for any positive rational number $\frac{k}{l}$, both k and l are obviously positive integers and $e^k = (e^{k/l})^l$, but if e^k is irrational, $e^{k/l}$ must also be irrational, because an integer power of a rational number must be rational, not irrational.

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As for the negative rational powers of e, it must be the reciprocal of the positive rational powers, because once all the positive rational powers of e are irrational, it is equivalent to proving that its negative rational powers are also irrational.

We assume that there exists some positive integer k. To make e^k a rational number, we can set $e^k = \frac{a}{b}$, and a and b are positive integers. Then we use the auxiliary function $f(x)$ from Section 2.2 to construct a new function satisfying $F(x)$.

$$F(x) = k^{2n} f(x) - k^{2n-1} f^{(1)}(x) + k^{2n-2} f^{(2)}(x) + \ldots\ldots + (-1)^i k^{2n-i} f^{(i)}(x) + \ldots\ldots + f^{(2n)}(x) + \ldots\ldots \quad (18)$$

Since $f(x)$ is a sub-polynomial, the function $F(x)$ is 0 for all terms after the term $f^{(2n)}(x)$, but it does not make a fundamental difference to continue adding up and writing it in the form of an infinite sum of terms. The function $F(x)$ so constructed has a feature that the form of the derivative function is somewhat similar to the original function, so it is easy to calculate to obtain.

$$F'(x) + kF(x) = k^{2n+1} f(x) \quad (19)$$

Based on equation 19 the differential equation can be constructed as follows.

$$\frac{\mathrm{d}}{\mathrm{d}x}[e^{kx} \cdot F(x)] = e^{kx} \cdot F'(x) + e^{kx} \cdot k \cdot F(x) = e^{kx} \cdot k^{2n+1} \cdot f(x) \quad (20)$$

Thus, we obtain the following integral equation.

$$L = b \cdot \int_0^1 e^{kx} \cdot k^{2n+1} \cdot f(x)\mathrm{d}x = b \cdot e^{kx} \cdot F(x)\Big|_0^1 = b \cdot e^k \cdot F(1) - b \cdot F(0) = a \cdot F(1) - b \cdot F(0) \quad (21)$$

According to Property III, $F(1)$ and $F(0)$ are integers, and a and b are also integers, thus L should be an integer. But on the other hand, according to property II, we have.

$$0 < L = b \cdot \int_0^1 e^{kx} \cdot k^{2n+1} \cdot f(x)\mathrm{d}x < b \cdot e^k \cdot k^{2n+1} \cdot \frac{1}{n!} = \frac{a \cdot k^{2n+1}}{n!} \quad (22)$$

At large values of n, $n!$ grows much faster than k^{2n+1}, so it is necessary to choose n large enough so that $n! > a \cdot k^{2n+1}$ and then get $0 < L < 1$, which contradicts that L is an integer. Thus e^k cannot be a rational number and the proof is over.

From the above process, we get the following conclusion: the natural exponent e itself is irrational, while any rational power of e (except 0) is also irrational.

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